MODELING, MODULARITY

and

MODULES

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Kolloquium T.U. Vienna

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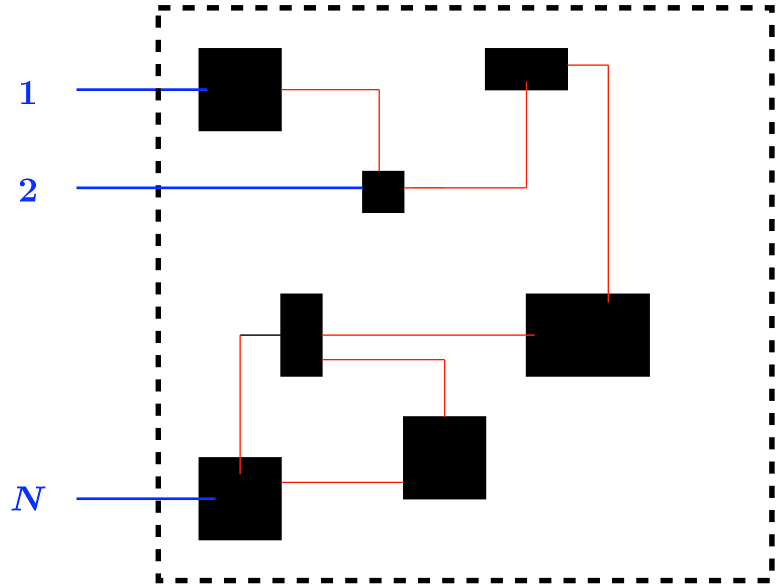
RuG

In honor of Inge Troch

on the occasion of her sixtieth birthday

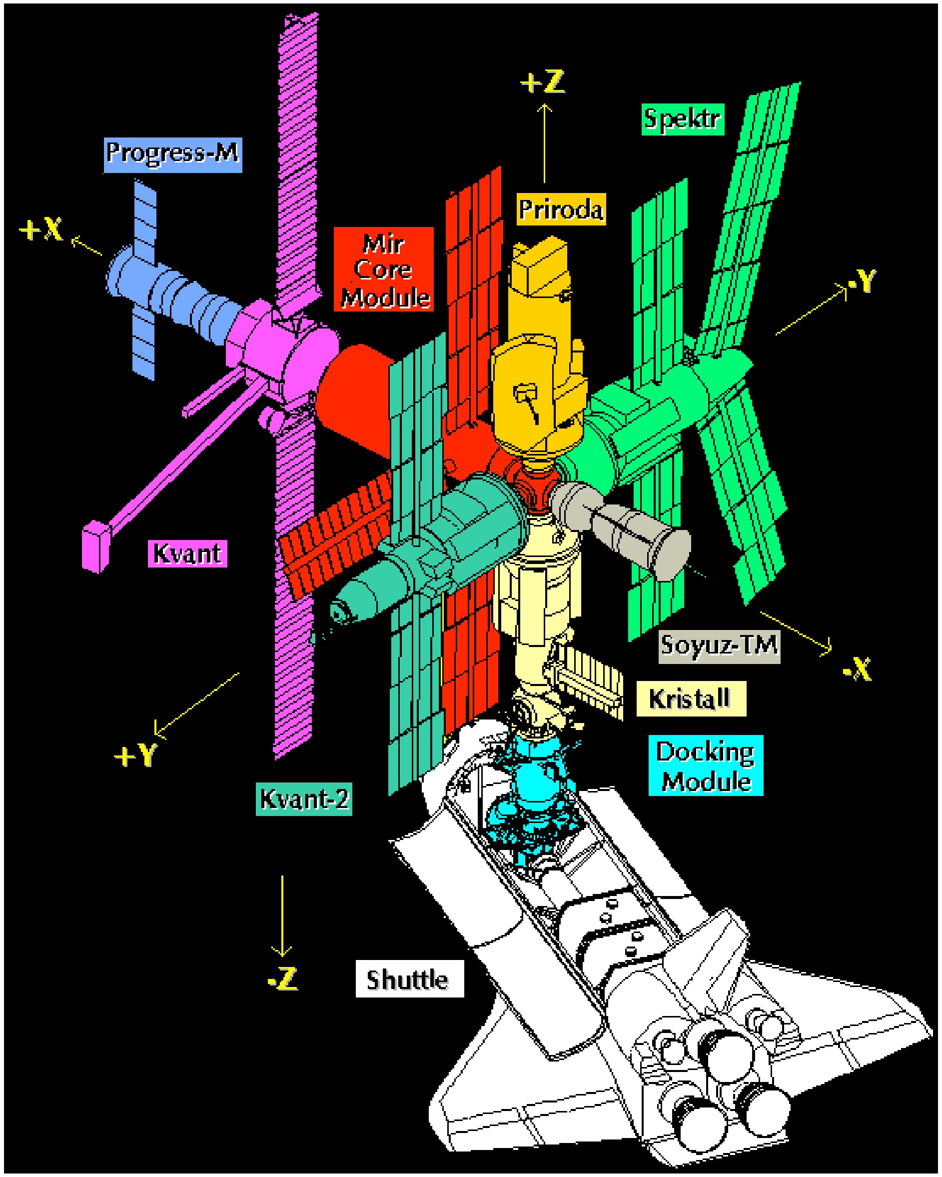
How do we model an interconnected system?



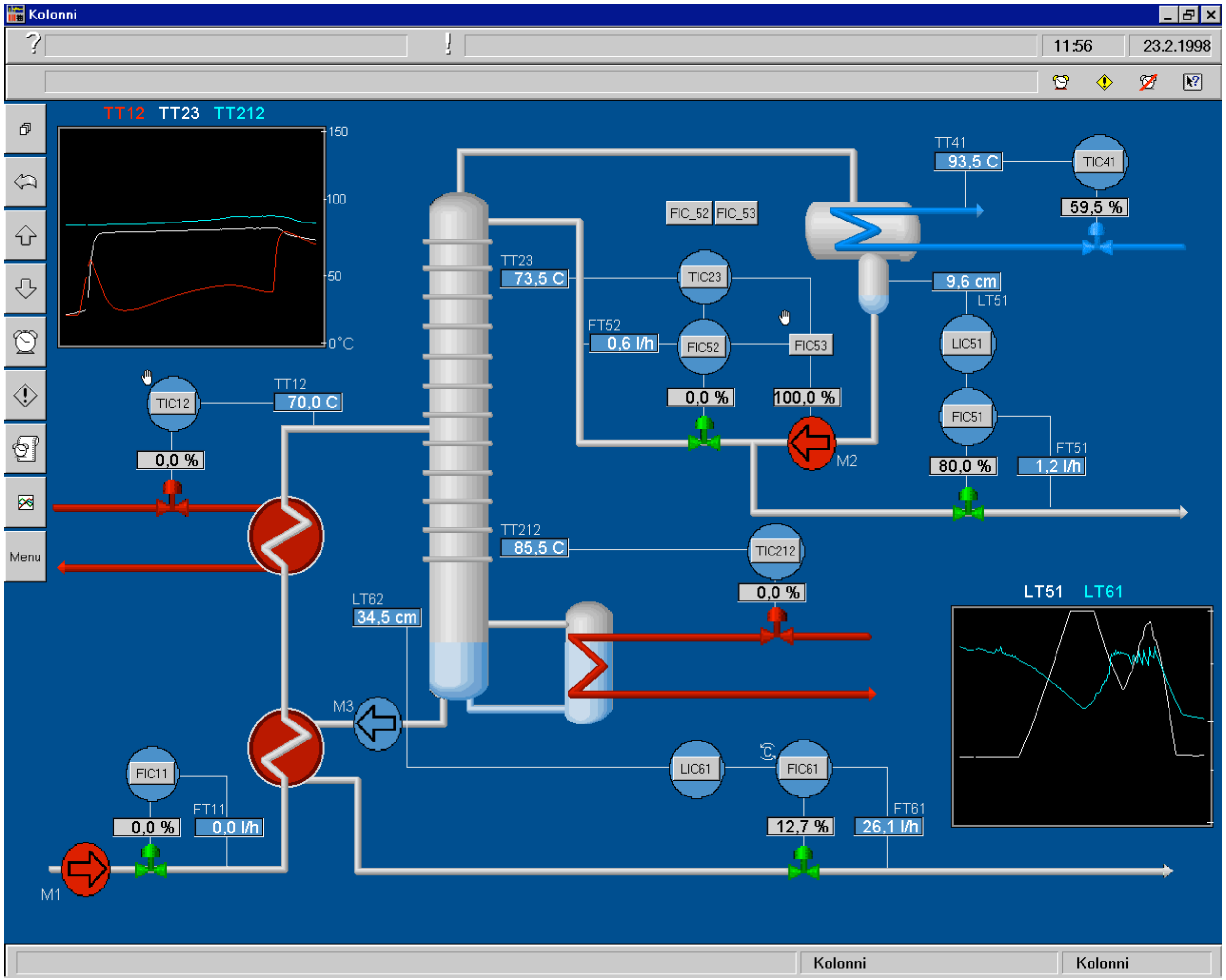
Interconnected system

Exs.: circuits, robots, chemical plants, etc. $\rightsquigarrow$ **Modularity**

$\rightsquigarrow$ **Object-oriented modeling**



+Z
Spektr
Progress-M
Priroda
+X
Mir Core Module
-Y
Kvant
Soyuz-TM
-X
Kristall
+Y
Docking Module
Kvant-2
-Z
Shuttle


Kolonni
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FT51
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TT212
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TIC212
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LT62
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LT51 LT61
M3
FIC11
FT11
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FT61
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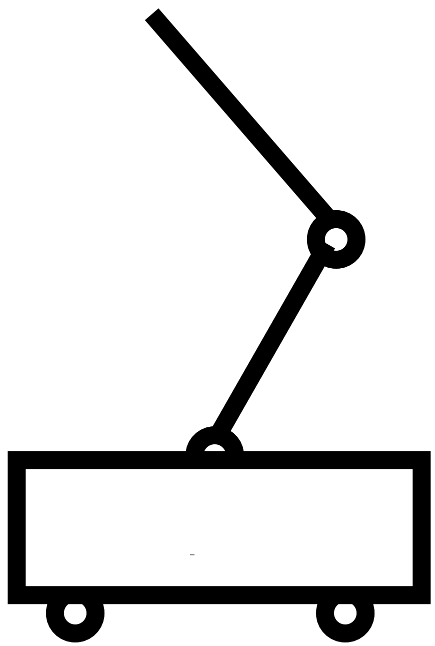
Kolonni

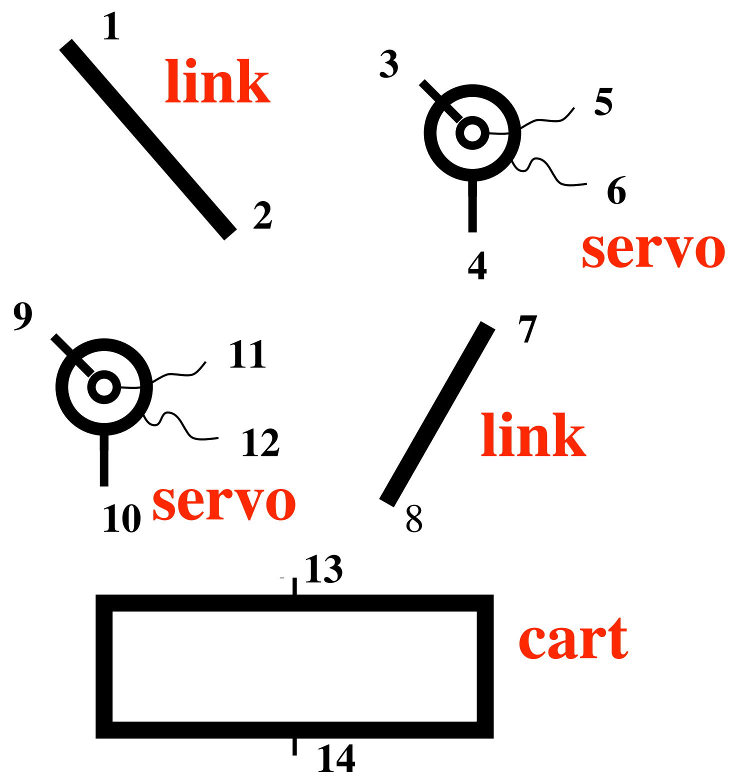
Example:



Cart with double pendulum

TEARING & ZOOMING

Tearing

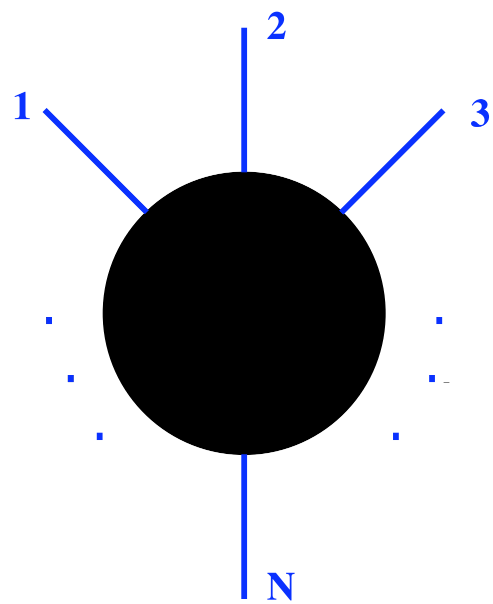


Zooming

Obtain models of the subsystems

Required modules in our example: **Solid bars, servo's.**

How do we model a building block?



Building block

Exs.: resistor, transformer, mass, spring, tank, solid bar, servo , etc.

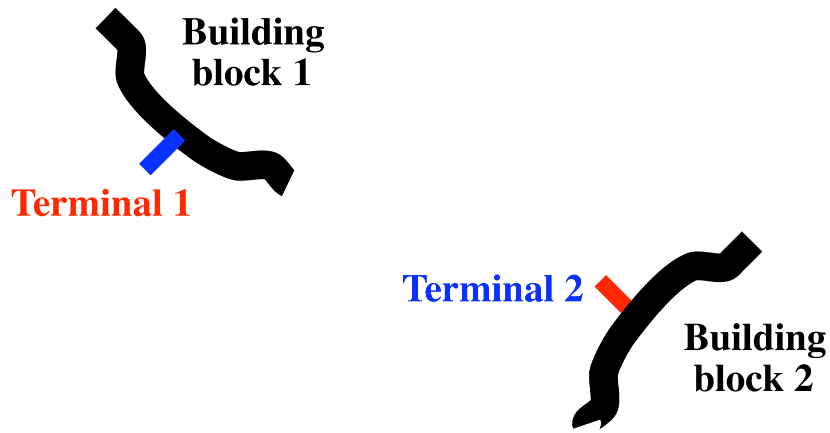
$\leadsto$ **Behavior of the terminal variables**

Examples of terminals and terminal variables:

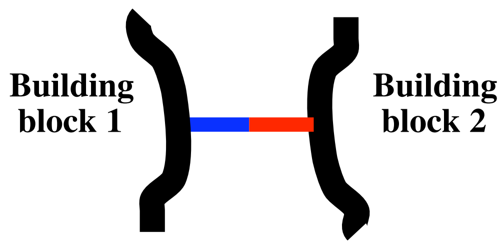
Type of terminal	Variables	Signal space
electrical	(voltage, current)	$\mathbb{R}^2$
mechanical (1-D)	(force, position)	$\mathbb{R}^2$
mechanical (2-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^1) \times (\mathbb{R}^2 \times T^* S^1)$
mechanical (3-D)	((position, attitude), (force, torque))	$(\mathbb{R}^2 \times S^2) \times (\mathbb{R}^2 \times T^* S^2)$
thermal	(temp., heat flow)	$\mathbb{R}^2$
fluidic	(pressure, flow)	$\mathbb{R}^2$
fluidic - thermal	(pressure, flow, temp., heat flow)	$\mathbb{R}^4$

How do we model an interconnection?

Before:



After:



Interconnection

$\rightsquigarrow$ **Identification of terminal variables**

Examples of interconnection equations:

Pair of terminals	Terminal 1	Terminal 2	Interconnection law
electrical	(V_1, I_1)	(V_2, I_2)	$V_1 = V_2, I_1 + I_2 = 0$
1-D mech.	(F_1, q_1)	(F_2, q_2)	$F_1 + F_2 = 0, q_1 = q_2$
2-D mech.			
thermal	(T_1, Q_1)	(T_2, Q_2)	$T_1 = T_2, Q_1 + Q_2 = 0$
fluidic	(p_1, f_1)	(p_2, f_2)	$p_1 = p_2, f_1 + f_2 = 0$
fluidic - thermal	(p_1, f_1, T_1, Q_1)	(p_2, f_2, T_2, Q_2)	$p_1 = p_2, f_1 + f_2 = 0,$ $T_1 = T_2, Q_1 + Q_2 = 0$

How do we approach such modeling tasks?

Approach should:

- be pedagogically convincing
- be computer oriented
- use mathematical language, with appropriate concepts
- aim at physical systems
- deal with interconnection without apologies
- be adapted to first principles models
- include dynamics, as well as space-time phenomena

Classical approaches

- **Dynamical systems, flows on manifolds:**

$$\frac{d}{dt}x = f(x), \quad [y = h(x)]$$

inadequate: does not cover open systems

- **input/output systems: more promising**

  aims at open systems, interconnections

- **bondgraphs: aims at physical interconnections**

  and energy considerations

Input/output systems

Building blocks:

- **input/output:**

  Recognize input and output variables ("cause and effect")

  Model the input-to-output map or relation

- **input/state/output:**

  Recognize input, output, and state variables

  Model the input-to-state and the state-to-output maps

$$\rightsquigarrow \quad \frac{d}{dt}x = f(x, u) \quad y = h(x)$$

Interconnections:

Identify inputs with outputs

Combine series and feedback connection ($\rightsquigarrow$ SIMULINK)

Beautiful concepts, very effective algorithms, but i/o is simply

not suitable as a 'first principles' starting point.

For building blocks:

Terminal variables are localized $\neq$ $\Rightarrow$ **System** $\Rightarrow$

A physical system is not a signal processor.

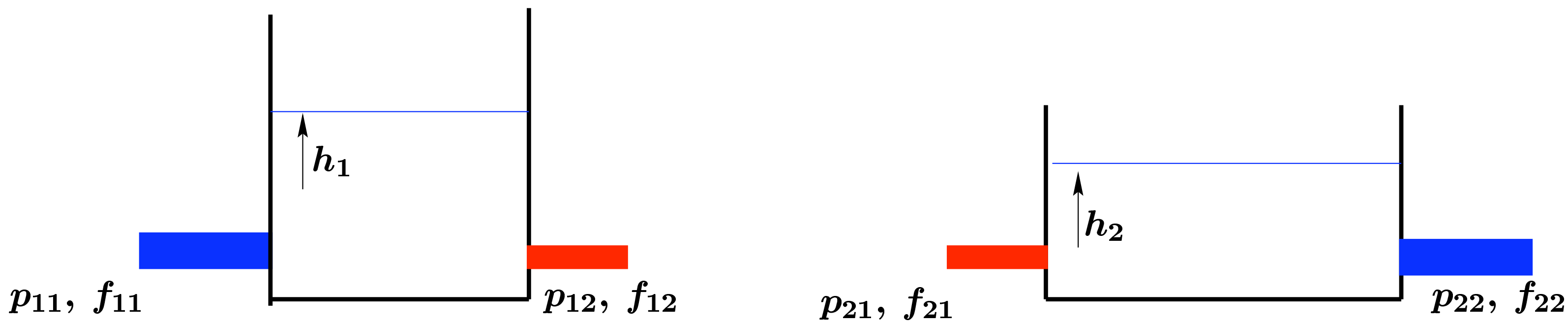
But: even CS and DES do not use the i/o approach!

For interconnected systems:

It is *not feasible to recognize the signal flow graph* before we have a model. The signal flow graph should be deduced from a model!

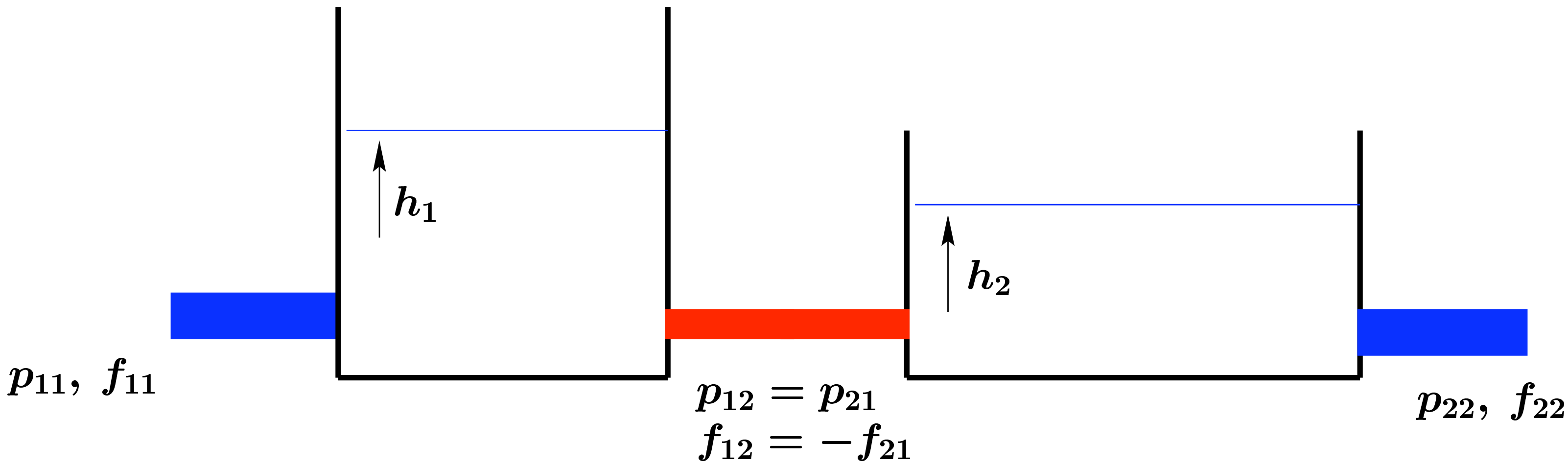
More suitable approach for dealing with interconnections $\leadsto$ **Bondgraphs.**

The inappropriateness of input - to - output connections is illustrated well by the following simple 2-tank physical example:



Logical choice of inputs: the pressures $p_{11}, p_{12}, p_{21}, p_{22}$,
and of outputs: the flows $f_{11}, f_{12}, f_{21}, f_{22}$
(h_1, h_2: state variables)

In any case, the input/output choice should be 'symmetric'.



Interconnection constraints:

$$p_{12} = p_{21}, \quad f_{12} = -f_{21}.$$

Equates two inputs and two outputs.

$\neq$ **equating inputs with outputs.**

A new set of concepts is needed !

BEHAVIORAL SYSTEMS

A system **:=** $\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T}$ $=$ **the set of *independent* variables**

time, space, time and space

$\mathbb{W}$ $=$ **the set of *dependent* variables**

(= where the variables take on their values),

signal space, space of field variables, . . .

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **:** ***the behavior*** **= the admissible trajectories**

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

for a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,

$w \notin \mathfrak{B}$: the model forbids the trajectory w.

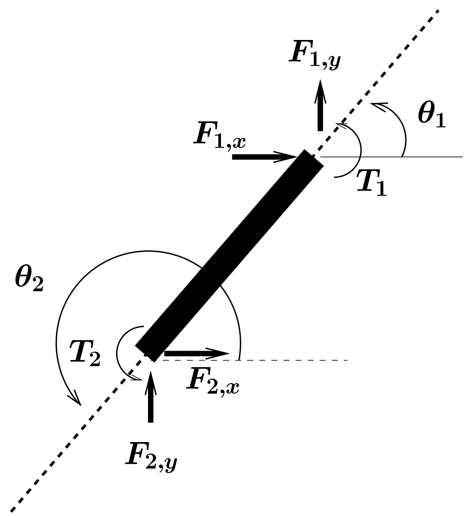
A system $\cong$ an exclusion law.

It tells what phenomena can happen, according to the model.

Usually: $\mathfrak{B}$ is specified as the set of solutions of a set of differential equations.

Use in modeling interconnected systems

Solid bar



Terminals: 2 mechanical 2-D terminals.

Variables: $x_1, y_1, \theta_1, x_2, y_2, \theta_2, F_{x_1}, F_{y_1}, T_1, F_{x_2}, F_{y_2}, T_2$.

Parameters: $L \in \mathbb{R}_+$ (length),
$m \in \mathbb{R}_+$ (mass per unit length).

Behavioral equations:

$$
\begin{aligned}
mL\frac{d^2}{dt^2}x_c &= F_{x_1} + F_{x_2},\\
mL\frac{d^2}{dt^2}y_c &= F_{y_1} + F_{y_2} - mLg,\\
m\frac{L^3}{12}\frac{d^2}{dt^2}\theta_c &= T_1 + T_2 - \frac{L}{2}F_{x_1}\sin(\theta_1)\\
&\quad + \frac{L}{2}F_{y_1}\cos(\theta_1) - \frac{L}{2}F_{x_2}\sin(\theta_2) + \frac{L}{2}F_{y_2}\cos(\theta_2),\\
\theta_1 &= \theta_c,\\
\theta_2 &= \theta_1 + \pi,\\
x_1 &= x_c + \frac{L}{2}\cos(\theta_c),\\
x_2 &= x_c - \frac{L}{2}\cos(\theta_c),\\
y_1 &= y_c + \frac{L}{2}\sin(\theta_c),\\
y_2 &= y_c - \frac{L}{2}\sin(\theta_c).
\end{aligned}
$$

Note: Contains latent variables x_c, y_c, θ_c.

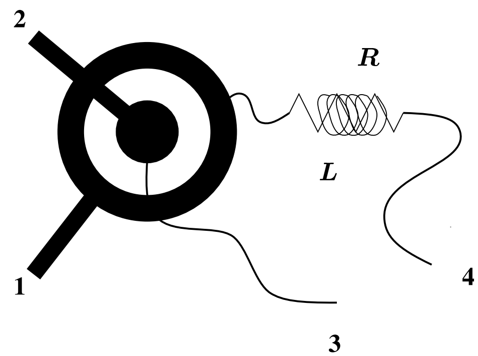
This defines a system with

$\mathbb{T} = \mathbb{R}$

$\mathbb{W} = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^*S^1) \times (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^*S^1)$

$\mathfrak{B}$ **= solutions** $(x_1, y_1, \theta_1, x_2, y_2, \theta_2, F_{x_1}, F_{y_1}, T_1, F_{x_2}, F_{y_2}, T_2)$
of the ODE's, suitably interpreted.

Hinge with servo



Terminals: 2 mechanical 2-D terminals, 2 electrical.

Variables: $(x_1, y_1, \theta_1, F_{x_1}, F_{y_1}, T_1,$
$x_2, y_2, \theta_2, F_{x_2}, F_{y_2}, T_2, V_3, I_3, V_4, I_4).$

Parameters: the rotor mass m_r, the stator mass m_s,
the rotor inertia J_r, the stator inertia J_s,
the inductance L, the resistance R of the motor circuit,
the motor torque constant K.

Behavioral equations:

$$(m_r + m_s)\frac{d^2}{dt^2}x_1 = F_{x_1} + F_{x_2}$$

$$(m_r + m_s)\frac{d^2}{dt^2}y_1 = F_{x_1} + F_{x_2}$$

$$J_r\frac{d^2}{dt^2}\theta_1 = T_1 + T_m$$

$$J_s\frac{d^2}{dt^2}\theta_2 = T_2 - T_m$$

$$V_3 - V_4 = L\frac{d}{dt}I_3 + RI_3 + K\frac{d}{dt}(\theta_1 - \theta_2)$$

$$KI_3 = T_m$$

$$x_1 = x_2$$

$$y_1 = y_2$$

$$I_3 = -I_4$$

Note: **The motor torque T_m is a latent variable.**

This defines a system with

$\mathbb{T} = \mathbb{R}$

$\mathbb{W} = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^2 \times T^*S^1)^2 \times (\mathbb{R}^2)^2$

$\mathfrak{B}$ **= solutions**

$(x_1, y_1, \theta_1, F_{x_1}, F_{y_1}, T_1, x_2, y_2, \theta_2, F_{x_2}, F_{y_2}, T_2, V_3, I_3, V_4, I_4)$

of the ODE's, suitably interpreted.

The cart with double pendulum

The list of the modules and the associated terminals:

Module	Type	Terminals	Parameters
Link 1	**bar**	**(7,8)**	L_1, m_1
Link 2	**bar**	**(1,2)**	L_2, m_2
Cart	**bar**	**(13,14)**	L_3, m_3
Servo 1	**servo**	**(9,10,11,12)**	$m_{r_1}, m_{s_1}, J_{r_1}, J_{r_1}, L_1, R_1, K_1$
Servo 2	**servo**	**(3,4,5,6)**	$m_{r_2}, m_{s_2}, J_{r_2}, J_{r_2}, L_2, R_2, K_2$

The interconnection architecture:

Pairing
$\{2, 3\}$
$\{4, 7\}$
$\{8, 9\}$
$\{10, 13\}$

Manifest variable assignment:

the variables on the external terminals $\{1, 5, 6, 11, 12, 14\}$.

Equations for the full behavior:

Equations of the modules:

$$m_1 L_1 \frac{d^2}{dt^2} x_{c_1} = F_{x_1} + F_{x_2},$$
$$m_1 L_1 \frac{d^2}{dt^2} y_{c_1} = F_{y_1} + F_{y_2} - m_1 L_1 g,$$
$$m_1 \frac{L_1^3}{12} \frac{d^2}{dt^2} \theta_{c_1} = T_1 + T_2$$
$$- \frac{L_1}{2} F_{x_1} \sin(\theta_1) + \frac{L_1}{2} F_{y_1} \cos(\theta_1) - \frac{L_1}{2} F_{x_2} \sin(\theta_2) + \frac{L_1}{2} F_{y_2} \cos(\theta_2),$$
$$\theta_1 = \theta_{c_1}, \theta_2 = \theta_1 + \pi,$$
$$x_1 = x_{c_1} + \frac{L_1}{2} \cos(\theta_{c_1}), x_2 = x_{c_1} - \frac{L_1}{2} \cos(\theta_{c_1}),$$
$$y_1 = y_{c_1} + \frac{L_1}{2} \sin(\theta_{c_1}), y_2 = y_{c_1} - \frac{L_1}{2} \sin(\theta_{c_1}),$$

$$m_2 L_2 \frac{d^2}{dt^2} x_{c_2} = F_{x_7} + F_{x_8},$$
$$m_2 L_2 \frac{d^2}{dt^2} y_{c_2} = F_{y_7} + F_{y_8} - m_2 L_2 g,$$
$$m_2 \frac{L_2^3}{12} \frac{d^2}{dt^2} \theta_{c_2} = T_7 + T_8$$
$$- \frac{L_2}{2} F_{x_7} \sin(\theta_7) + \frac{L_2}{2} F_{y_7} \cos(\theta_7) - \frac{L_2}{2} F_{x_8} \sin(\theta_8) + \frac{L_2}{2} F_{y_8} \cos(\theta_8),$$
$$\theta_7 = \theta_{c_2}, \theta_8 = \theta_7 + \pi,$$
$$x_7 = x_{c_2} + \frac{L_1}{2} \cos(\theta_{c_2}), x_8 = x_{c_2} - \frac{L_1}{2} \cos(\theta_{c_2}),$$
$$y_7 = y_{c_2} + \frac{L_1}{2} \sin(\theta_{c_2}), y_8 = y_{c_2} - \frac{L_1}{2} \sin(\theta_{c_2}),$$

$$m_3 L_3 \frac{d^2}{dt^2} x_{c_3} = F_{x_{13}} + F_{x_{14}},$$
$$m_3 L_3 \frac{d^2}{dt^2} y_{c_3} = F_{y_{13}} + F_{y_{14}} - m_3 L_3 g,$$
$$m_3 \frac{L_3^3}{12} \frac{d^2}{dt^2} \theta_{c_3} = T_{13} + T_{14}$$
$$- \frac{L_3}{2} F_{x_{13}} \sin(\theta_{13}) + \frac{L_3}{2} F_{y_{13}} \cos(\theta_{13}) - \frac{L_3}{2} F_{x_{14}} \sin(\theta_{14}) + \frac{L_3}{2} F_{y_{14}} \cos(\theta_{14}),$$
$$\theta_{13} = \theta_{c_3}, \theta_{14} = \theta_{c_3} + \pi,$$
$$x_{13} = x_{c_3} + \frac{L_1}{2} \cos(\theta_{c_3}),$$
$$x_{14} = x_{c_3} - \frac{L_1}{2} \cos(\theta_{c_3}), y_{13} = y_{c_3} + \frac{L_1}{2} \sin(\theta_{c_3}),$$
$$y_{14} = y_{c_3} - \frac{L_1}{2} \sin(\theta_{c_3}),$$

$$(m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} x_3 = F_{x_3} + F_{x_4},$$
$$(m_{r_1} + m_{s_1}) \frac{d^2}{dt^2} y_3 = F_{y_3} + F_{y_4},$$
$$J_{r_1} \frac{d^2}{dt^2} \theta_3 = T_3 + T_m,$$
$$J_{s_1} \frac{d^2}{dt^2} \theta_4 = T_4 - T_m,$$
$$V_5 - V_6 = L_1 \frac{d}{dt} I_5 + R_1 I_5 + K \frac{d}{dt} (\theta_3 - \theta_4),$$
$$K_1 I_5 = T_{m_1}, x_3 = x_4, y_3 = y_4, I_5 = -I_6,$$

$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}x_9 = F_{x_9} + F_{x_{10}},$$
$$(m_{r_2} + m_{s_2})\frac{d^2}{dt^2}y_9 = F_{y_9} + F_{y_{10}},$$
$$J_{r_2}\frac{d^2}{dt^2}\theta_9 = T_9 + T_m,$$
$$J_{s_2}\frac{d^2}{dt^2}\theta_{10} = T_{10} - T_m,$$
$$V_{11} - V_{12} = L_2\frac{d}{dt}I_{11} + R_2 I_{11} + K\frac{d}{dt}(\theta_9 - \theta_{10}),$$
$$K_2 I_{11} = T_{m_2}, x_{10} = x_{11}, y_{10} = y_{11}, I_{11} = -I_{12},$$

Interconnection equations:

$$F_{x_2} + F_{x_3} = 0,\ F_{y_2} + F_{y_3} = 0,\ x_2 = x_3,\ y_2 = y_3,\ \theta_2 = \theta_3 + \pi,\ T_2 + T_3 = 0,$$

$$F_{x_4} + F_{x_7} = 0,\ F_{y_4} + F_{y_7} = 0,\ x_4 = x_7,\ y_4 = y_7,\ \theta_4 = \theta_7 + \pi,\ T_4 + T_7 = 0,$$

$$F_{x_8} + F_{x_9} = 0,\ F_{y_8} + F_{y_9} = 0,\ x_8 = x_9,\ y_8 = y_9,\ \theta_8 = \theta_9 + \pi,\ T_8 + T_9 = 0,$$

$$F_{x_{10}} + F_{x_{13}} = 0,\ F_{x_{10}} + F_{x_{13}} = 0, x_{10} = x_{13},\ y_{10} = y_{13},$$
$$\theta_{10} = \theta_{13} + \pi,\ T_{10} + T_{13} = 0.$$

The behavioral approach lends itself also to some

real mathematics

Notation:

Ring of real polynomials in $\mathtt{n}$ variables $\rightsquigarrow \mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

$\mathbb{R}^{\mathtt{n}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, $\mathbb{R}^{\bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, $\mathbb{R}^{\mathtt{n}_1 \times \mathtt{n}_2}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$\mathbb{R}^{\bullet \times \mathtt{n}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, $\mathbb{R}^{\mathtt{n} \times \bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$\mathbb{R}^{\bullet \times \bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

In the remainder of this lecture, we consider systems

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

with

$$\mathbb{T} = \mathbb{R}^{n}, \quad \mathbb{W} = \mathbb{R}^{\mathtt{w}},$$

$$w : \mathbb{R}^{n} \to \mathbb{R}^{\mathtt{w}}, (w_1(x_1, \cdots, x_{\mathtt{n}}), \cdots, w_{\mathtt{w}}(x_1, \cdots, x_{\mathtt{n}})),$$

often, $\mathtt{n} = 1$**, independent variable time,**

or $\mathtt{n} = 4$**, independent variables** (t, x, y, z)**,**

$\mathfrak{B}$ **= solutions of a system of constant coefficient linear ODE's or PDE's.**

Linear differential systems (PDE's)

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathtt{n}$ **independent variables,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**

$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

Define its behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))$$

$\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**

Notation:

$$(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}},$$

$$\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})).$$

'kernel representation'.

An example: ***Maxwell's equations***

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0\,, \\ c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}. \end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B} \in \mathfrak{L}_4^8$ **: the set of solutions to Maxwell's equations.**

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

Three representation results

- **Relation with sub-modules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**
- **Elimination theorem**
- **Controllability and image representations**

R defines $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_\mathtt{n}}))$, but not vice-versa!

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_\mathtt{n}^\mathtt{w}$??

Define the *annihilators* of $\mathfrak{B}$ by

$$\mathfrak{N}_\mathfrak{B} := \{n \in \mathbb{R}^\mathtt{w}[\xi_1, \cdots, \xi_\mathtt{n}] \mid n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_\mathtt{n}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_\mathfrak{B}$ is clearly a sub-module of $\mathbb{R}^\mathtt{w}[\xi_1, \cdots, \xi_\mathtt{n}]$.

Let $< R >$ denote the sub-module of $\mathbb{R}^\mathtt{w}[\xi_1, \cdots, \xi_\mathtt{n}]$ spanned by the transposes of the rows of R. Obviously $< R > \subseteq \mathfrak{N}_\mathfrak{B}$. But, in fact:

$$\mathfrak{N}_\mathfrak{B} = < R >$$

Therefore

$$\mathfrak{L}_\mathtt{n}^\mathtt{w} \xleftrightarrow{1:1} \text{sub-modules of } \mathbb{R}^\mathtt{w}[\xi_1, \cdots, \xi_\mathtt{n}]$$

Elimination

First principles modeling usually requires auxiliary variables (state variables, interconnection variables, etc.). This invariably leads (perhaps after linearization) to modeling equations of the form:

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell} \quad (**)$$

$w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ ***'manifest' variables,***
$\ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell})$ ***'latent' variables,***
R **and** M **suitably sized polynomial matrices in** $\mathtt{n}$ **variables.**

We view $(**)$ as a model for the behavior of the variables w.

Define the *manifest behavior* of $(**)$ as

$$\mathfrak{B} = \{ w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell}) : (**) \text{ holds } \}$$

i.e., $$\mathfrak{B} = (R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathrm{n}}}))^{-1} M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathrm{n}}}) \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\ell}).$$

Does $\mathfrak{B}$ belong to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$?

Theorem: **It does!**

Proof: **Fundamental principle.**

Algorithm: **Syzygies, Gröbner bases, computer algebra.**

Which PDE's describe $(\vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$, ρ from Maxwell's equations. Straightforward computation of the relevant left syzygy yields

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Elimination theorem $\Rightarrow$ this exercise would be exact & successful.

Controllability

Definition: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is said to be**

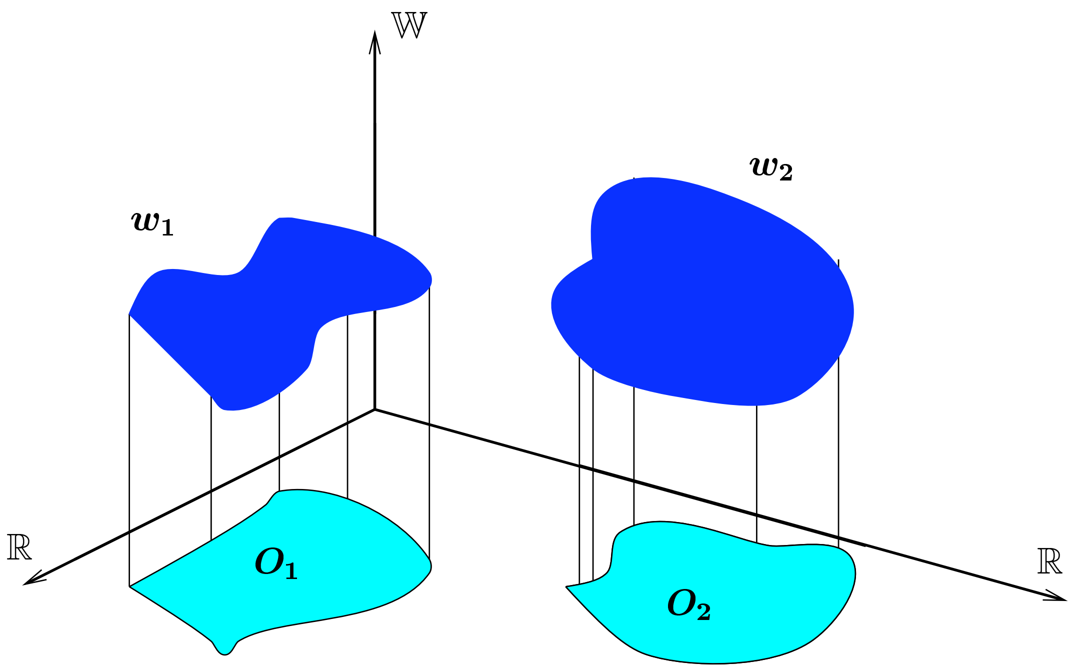
controllable

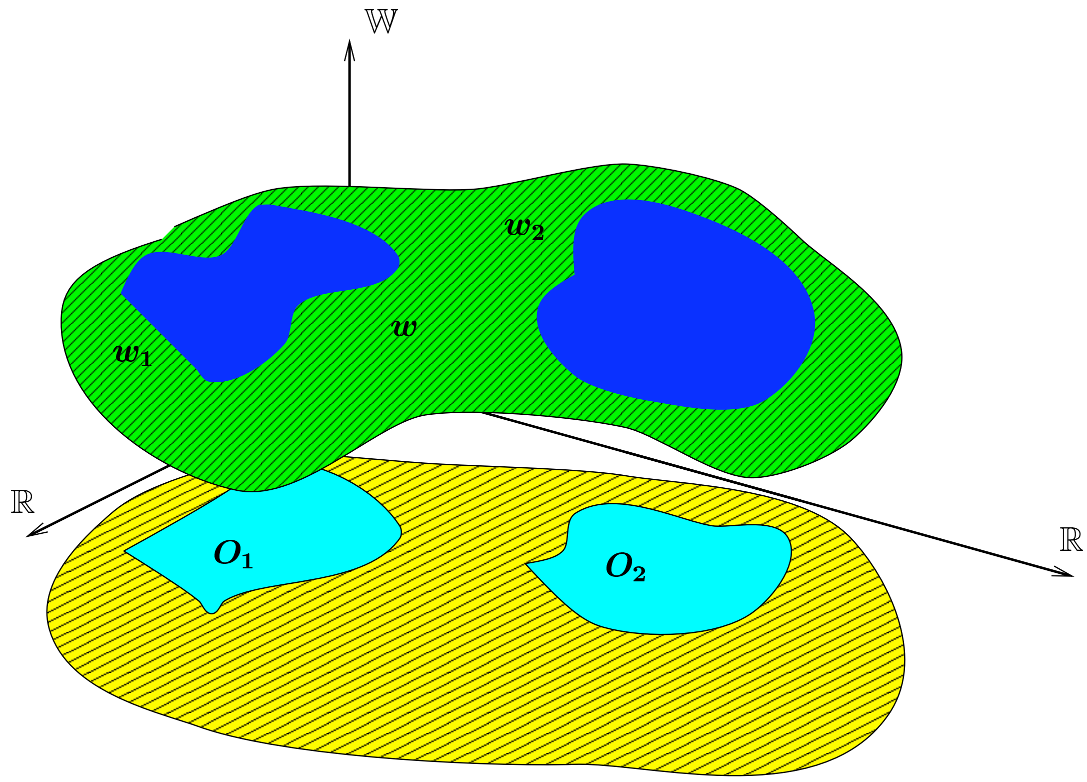
if for all $w_1, w_2 \in \mathfrak{B}$ and
for all $O_1, O_2 \subset \mathbb{R}^n$, non-overlapping closure,
there exists $w \in \mathfrak{B}$ such that $w|_{O_1} = w_1|_{O_1}$ and $w|_{O_2} = w_2|_{O_2}$.

Controllability :$\Leftrightarrow$ the elements of $\mathfrak{B}$ are 'patch-able'.

Special case: Kalman controllability for input/state systems.

In pictures:





𝕎
w_2
w
w_1
ℝ
ℝ
O_1
O_2

Conditions for controllability

Representations of $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

$$R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})w = 0 \qquad (*)$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}}))$;

$$R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})\ell \qquad (**)$$

called a *'latent variable' representation* of the manifest behavior $\mathfrak{B} = (R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}}))^{-1}M(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}^{\ell})$.

Missing link:

$$w = M(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}})\ell \qquad (***)$$

called an *'image' representation* of $\mathfrak{B} = \mathrm{im}(M(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathrm{n}}}))$.

Elimination theorem $\Rightarrow$

every image (of a linear constant coefficient PDO) is also a kernel.

¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

1. $\mathfrak{B}$ **is controllable,**
2. $\mathfrak{B}$ **admits an image representation,**
3. **for any** $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$$a^\top[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ **is torsion free,**

etc.

Algorithm: R + syzygies + Gröbner basis $\Rightarrow$ numerical test on coefficients of R.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

Summary

- **The i/s/o paradigm is inadequate for first principles *modeling.* It fails in the first examples, it is unsuited for interconnection, for *modularity*, for object-oriented modeling.**
- **Universal paradigm: *BEHAVIORAL SYSTEMS*.**
- **$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ closed under intersection, addition, and projection.**
- **Linear shift-invariant differential systems**
  $\overset{1:1}{\longleftrightarrow}$ ***sub-modules* of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.**
- **Controllability $\Leftrightarrow$ *sub-module* is torsion-free.**
- **$\exists$ extensive theory, adapted to modeling, covering all the classical results, unifying physical models with DES, etc.**

More information?

Surf to

`http://www.math.rug.nl/∽willems`

THANK YOU

&

BEST WISHES TO YOU, INGE !